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Analysis of two parallel symmetric cracks using the non-local theory

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1. Introduction

In papers[1-4], the state of stress near the tip of a sharp line crack in an elastic plate subjected to uniform tension, in-plane shear and anti-plane shear are discussed. The field equations employed in the solutions of these problems are those of the theory of the non-local elasticity. The solutions gave finite stress at the crack tips, thus resolving a fundamental problem that has remained unsolved over half a century. This enabled us to employ the maximum-stress hypothesis to deal with fracture problem and the composite materials problem in a natural way. However, they were not exact and there is oscillatory stress near the crack tip[1]. The iteration error is also not reasonable[1-4], because the dual integral equation has a super singularity integral kernel. To overcome the difficulty, the Schmidt method[5] will be used. Recently, the same problems in the papers [1-4] have been resolved in papers[6-8] by using the Schmidt method and the results are more accurate and more reasonable. In papers[9-10], the problems for a crack or two cracks were investigated by using the non-local theory. To the author's knowledge, analytical treatment of two parallel symmetric cracks problem by using the non-local theory has not been attempted.
For the above-mentioned reasons, the present paper deals with the problem of two parallel symmetric cracks in an elastic plate by using the non-local theory. For overcoming the mathematical difficulties, one has to accept some assumptions as Nowinski[11-12], one-dimensional non-local kernel function is used instead of two-dimensional kernel function for the anti-plane

problem to obtain the stress occur at the crack tips. Certainly, the assumption should be further investigated to satisfy the realistic condition. The Fourier transform is applied and a mixed boundary value problem is formulated. Then a set of dual integral equations is solved with the Schmidt method[5]. In solving the equations, the gaps of the displacement along the crack surface are expanded in a series of Jacobi polynomials. This process is quite different from that adopted in Eringen's works[1-4]. The solution, as expected, does not contain the stress singularity near the crack tips. The stress field along the crack line depends not only on the crack length, the distance between two parallel cracks, but also on the lattice parameter.

2. Basic Equations of Non-local Elasticity

Basic equations of linear, homogeneous, isotropic, non-local elastic solids, with vanishing body force are

$$\tau_{kl,k} = 0, \qquad \tau_{kl} = \int_V \alpha(|X'-X|)\sigma_{kl}(X')\, dV(X') \tag{1}$$

$$\sigma_{ij}(X') = \lambda u_{r,r}(X')\delta_{ij} + \mu[u_{i,j}(X') + u_{j,i}(X')] \tag{2}$$

where the only difference from classical elasticity is in the stress constitutive equations (1) in which the stress $\tau_{kl}(X)$ at a point X depends on the strains $e_{kl}(X')$, at all points of the body. For homogeneous and isotropic solids there exist only two material constants, λ and μ are the Lame constants of classical elasticity. $\alpha(|X'-X|)$ is known as influence function, and is the function of the distance $|X'-X|$. The expression (2) is the classical Hook's law. Substitution of equation (2) into equation (1) and using Green-Gauss theorem, it can be obtained:

$$\int_V \alpha(|X'-X|)[(\lambda+\mu)u_{k,kl}(X') + \mu u_{l,kk}(X')]dV(X') - \int_{\partial V} \alpha(|X'-X|)\sigma_{kl}(X')da_k(X') = 0 \tag{3}$$

Here the surface integral may be dropped if the only surface of the body is at infinity.

3. The Crack Model

It is assumed that there are two parallel symmetric cracks of length $2l$ in an elastic plate as shown in Fig.1. h is the distance between the two cracks. As discussed in [1-4], when the crack is subjected to the anti-plane shear stress τ_0, the boundary conditions on the crack faces at $y=0$ are:

$$w^{(1)}(x,h) = w^{(2)}(x,h), \quad \tau_{yz}^{(1)}(x,h) = \tau_{yz}^{(2)}(x,h), \quad |x| > l \tag{4}$$

$$w^{(2)}(x,0) = w^{(3)}(x,0), \quad \tau_{yz}^{(2)}(x,0) = \tau_{yz}^{(3)}(x,0), \quad |x| > l \tag{5}$$

$$\tau_{yz}^{(1)}(x,h) = \tau_{yz}^{(2)}(x,h) = -\tau_0, \quad \tau_{yz}^{(2)}(x,0) = \tau_{yz}^{(3)}(x,0) = -\tau_0 \quad |x| \le l \tag{6}$$

$$w^{(1)}(x,y)=w^{(2)}(x,y)=w^{(3)}(x,y)=0 \quad ,(x^2+y^2)^{1/2}\to\infty \tag{7}$$

Note that all quantities with superscript k (k=1, 2,3) refer to the upper half plane 1, the layer 2 and the lower half plane 3 as in figure 1, respectively.

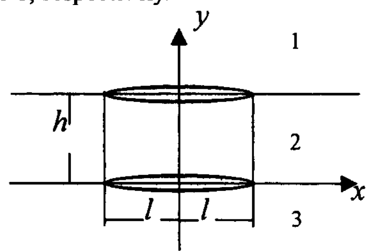

Fig.1. Two parallel symmetric cracks in the plane

4. The Dual Integral Equations and the Solution

According to the boundary conditions, the equation (3) can be written as follow:

$$\mu\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha(|x'-x|,|y'-y|)\nabla^2 w(x',y')dx'dy'$$
$$-\int_{-l}^{l}\alpha(|x'-x|,0)\,[\![\sigma_{yz}(x',0)]\!]\,dx'-\int_{-l}^{l}\alpha(|x'-x|,h)\,[\![\sigma_{yz}(x',h)]\!]\,dx'=0 \tag{8}$$

where $[\![\sigma_{yz}(x,y)]\!]=\sigma_{yz}(x,y^+)-\sigma_{yz}(x,y^-)$ are a jump across the crack.

From the works[2, 4], it can be obtained:

$$[\![\sigma_{yz}(x,0)]\!]=[\![\sigma_{yz}(x,h)]\!]=0 \qquad \text{for all } x \tag{9}$$

Define the Fourier transform by the equations

$$\bar{f}(s)=\int_{-\infty}^{\infty}f(x)e^{-isx}dx \quad , \quad f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(s)e^{isx}ds \tag{10}$$

For solving the problem, the Fourier transform of equation (8) with respect x can be given as follows:

$$\mu\int_{-\infty}^{\infty}\bar{\alpha}(|s|,|y'-y|)[(-s^2)\bar{w}+\frac{\partial^2\bar{w}}{\partial y^2}]dy'=0 \tag{11}$$

From (11), we can derive

$$\frac{\partial^2\bar{w}}{\partial y^2}-s^2\bar{w}=0 \tag{12}$$

whose solutions do not present difficulties, we have

$$\bar{w}^{(1)}(s,y)=A_1(s)e^{-sy},\,(y\ge h) \tag{13}$$

$$\overline{w}^{(2)}(s,y) = A_2(s)e^{-sy} + B_2(s)e^{sy}, \; (h \ge y \ge 0) \tag{14}$$

$$\overline{w}^{(3)}(s,y) = A_3(s)e^{sy}, \; (y \le 0) \tag{15}$$

where $A_1(s)$, $A_2(s)$, $B_2(s)$ and $A_3(s)$ are to be determined from the boundary conditions. The stress field, according to (1-2), is given by:

$$\begin{aligned}\tau_{yz}(x,y) = &-\frac{2\mu}{\pi}\int_0^\infty sA_1(s)ds\int_h^\infty dy'\int_{-\infty}^\infty [\alpha(|x'-x|,|y'-y|)e^{-sy'}\cos(sx')dx' \\ &-\frac{2\mu}{\pi}\int_0^\infty sA_2(s)ds\int_0^h dy'\int_{-\infty}^\infty [\alpha(|x'-x|,|y'-y|)e^{-sy'}\cos(sx')dx' \\ &+\frac{2\mu}{\pi}\int_0^\infty sB_2(s)ds\int_0^h dy'\int_{-\infty}^\infty [\alpha(|x'-x|,|y'-y|)e^{sy'}\cos(sx')dx' \\ &+\frac{2\mu}{\pi}\int_0^\infty sA_3(s)ds\int_{-\infty}^0 dy'\int_{-\infty}^\infty [\alpha(|x'-x|,|y'-y|)e^{sy'}\cos(sx')dx'\end{aligned} \tag{16}$$

What now remains is to solve the function w by using the equation (16) and the boundary conditions. It seems obvious that a rigorous solution of such a problem encounters serious if not unsurmountable mathematical difficulties, and one has to resort to an approximate procedure. In the given problem, according to the assumptions of Nowinski's[11-12], the non-local interaction in y direction can be ignored. In view of our assumptions, it can be given

$$\alpha(|x'-x|,|y'-y|) = \alpha_0(|x'-x|)\delta(y'-y) \tag{17}$$

As discussed in [4, 11, 12], it was taken

$$\alpha_0 = \chi_0 \exp(-(\beta/a)^2(x'-x)^2), \text{ with } \quad \chi_0 = \frac{1}{\sqrt{\pi}}\beta/a \tag{18}$$

where β is a constant (here $\beta = e_0\sqrt{\pi}/l$, e_0 is a constant appropriate to each material), a is the lattice parameter. So it can be obtained

$$\overline{\alpha}_0(s) = \exp(-\frac{(sa)^2}{(2\beta)^2}) \tag{19}$$

$\overline{\alpha}_0(s) = 1$ for the limit $a \to 0$, so that the equation (16) reverts to the well-known equation of the classical theory. So from the equation (16), we have

$$\tau_{yz}^{(1)}(x,h) = -\frac{2\mu}{\pi}\int_0^\infty \exp(-\frac{a^2s^2}{4\beta^2})\exp(-sh)sA_1(s)\cos(sx)ds \tag{20}$$

$$\tau_{yz}^{(2)}(x,0) = -\frac{2\mu}{\pi}\int_0^\infty \exp(-\frac{a^2s^2}{4\beta^2})s[A_2(s) - B_2(s)]\cos(sx)ds \tag{21}$$

$$\tau_{yz}^{(3)}(x,0)=\frac{2\mu}{\pi}\int_0^\infty \exp(-\frac{a^2s^2}{4\beta^2})sA_3(s)\cos(sx)ds \tag{22}$$

For solving the problem, the gap functions of the crack surface displacements is defined as follows:

$$f_1(x)=w^{(1)}(x,h^+)-w^{(2)}(x,h^-),\ \ f_2(x)=w^{(2)}(x,0^+)-w^{(3)}(x,0^-) \tag{23}$$

Substituting equations (14-15) into equations (23), and applying the Fourier transform, it can be obtained

$$\bar{f}_1(s)=[A_1(s)-A_2(s)]e^{-sh}-B_2(s)e^{sh},\ \ \bar{f}_2(s)=A_2(s)+B_2(s)-A_3(s) \tag{24}$$

Substituting equations (20-22) into equations (4-6), it can be obtained

$$[A_1(s)-A_2(s)]e^{-2sh}=-B_2(s),\ \ A_2(s)-B_2(s)=-A_3(s) \tag{25}$$

By solving four equations (24-25) with four unknown functions $A_1(s)$, $A_2(s)$, $B_2(s)$ and $A_3(s)$ and applying the boundary conditions (4-6), it can be obtained:

$$\int_0^\infty \frac{1}{2}\exp(-\frac{a^2s^2}{4\beta^2})s[\bar{f}_1(s)+\exp(-sh)\bar{f}_2(s)]\cos(sx)ds=\frac{\pi\tau_0}{2\mu}\ ,\ \ |x|\le l \tag{26}$$

$$\int_0^\infty \frac{1}{2}\exp(-\frac{a^2s^2}{4\beta^2})s[\exp(-sh)\bar{f}_1(s)+\bar{f}_2(s)]\cos(sx)ds=\frac{\pi\tau_0}{2\mu}\ ,\ \ |x|\le l \tag{27}$$

$$\int_0^\infty \bar{f}_1(s)\cos(sx)ds=0\ ,\ \ \int_0^\infty \bar{f}_2(s)\cos(sx)ds=0,\ \ |x|>l \tag{28}$$

From the (26-28), it can be obtained

$$\bar{f}_1(s)=\bar{f}_2(s)\Rightarrow f_1(x)=f_2(x),\ \ \tau_{yz}^{(1)}(x,h)=\tau_{yz}^{(2)}(x,h)=\tau_{yz}^{(2)}(x,0)=\tau_{yz}^{(3)}(x,0)=\tau_{yz} \tag{29}$$

Here we just solve the dual integral equation (26) and (28). Since the only difference between the classical and the non-local equations is in the introduction of the function $\exp(-\frac{a^2s^2}{4\beta^2})$, it is logical to utilize the classical solution to convert the system (26-28) to an integral equation of the second kind which is generally better behaved. For $a=0$, then the equations (26-28) reduce to the dual integral equations for same problem in classical elasticity. To determine the unknown functions $\bar{f}_1(s)$ and $\bar{f}_2(s)$, the dual-integral equations (26-28) must be solved. The dual integral equations can be considered to be a single integral equation of the first kind with a discontinuous kernel[1]. It is well-known in the literature that integral equations of the first kind are generally ill-posed in sense of Hadamard, i.e. small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. In this paper, the Schmidt method was used to overcome the difficulty. The gap functions of the crack surface displacement are be represented by the following series:

$$f_1(x) = f_2(x) = \sum_{n=1}^{\infty} a_n P_{2n-2}^{(\frac{1}{2},\frac{1}{2})}(\frac{x}{l})(1-\frac{x^2}{l^2})^{\frac{1}{2}}, \quad \text{for} -l \le x \le l, y=0 \tag{30}$$

where a_n is unknown coefficients to be determined and $P_n^{(\frac{1}{2},\frac{1}{2})}(x)$ is a Jacobi polynomial[13]. The Fourier transformation of equation (30) are:

$$\bar{f}_1(s) = \sum_{n=1}^{\infty} a_n G_n \frac{1}{s} J_{2n-1}(sl), \quad G_n = 2\sqrt{\pi}(-1)^{n-1}\frac{\Gamma(2n-\frac{1}{2})}{(2n-2)!} \tag{31}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting equation (31) into equations (26-28), respectively, the equations (28) has been automatically satisfied, the equation (26) reduces to the form for $-l < x < l$ (the equation (27) can be solved similar as equation (26)),

$$\sum_{n=1}^{\infty} a_n G_n \int_0^{\infty} \frac{1}{2} \exp(-\frac{a^2 s^2}{4\beta^2})[\exp(-sh)+1]J_{2n-1}(sl)\cos(sx)ds = \frac{\pi\tau_0}{2\mu} \tag{32}$$

For a large s, the integrands of the equation (32) are almost decreases exponentially. So they can be evaluated numerically by Filon's method[14]. Equation (32) can now be solved for the coefficients a_n by the Schmidt method[5]. For brevity, the equation (32) can be rewritten as

$$\sum_{n=1}^{\infty} a_n E_n(x) = U(x), -l < x < l \tag{33}$$

where $E_n(x)$ and $U(x)$ are known functions and the coefficients a_n are to be determined. A set of functions $P_n(x)$ which satisfy the orthogonality condition

$$\int_{-l}^{l} P_m(x)P_n(x)dx = N_n \delta_{mn}, \qquad N_n = \int_{-l}^{l} P_n^2(x)dx \tag{39}$$

can be constructed from the function, $E_n(x)$, such that

$$P_n(x) = \sum_{i=1}^{n} \frac{M_{in}}{M_{nn}} E_i(x) \tag{34}$$

where M_{ij} is the cofactor of the element d_{ij} of D_n, which is defined as

$$D_n = \begin{bmatrix} d_{11}, d_{12}, d_{13}, \ldots, d_{1n} \\ d_{21}, d_{22}, d_{23}, \ldots, d_{2n} \\ d_{31}, d_{32}, d_{33}, \ldots, d_{3n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ d_{n1}, d_{n2}, d_{n3}, \ldots, d_{nn} \end{bmatrix}, \quad d_{ij} = \int_{-l}^{l} E_i(x) E_j(x) dx \tag{35}$$

Using equations (32-35), we obtain

$$a_n = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \quad \text{with} \quad q_j = \frac{1}{N_j} \int_{-l}^{l} U(x) P_j(x) dx \tag{36}$$

5. Numerical Calculations and Discussion

From the works[6, 7, 8, 9, 10, 15], it can be seen that the Schmidt method is performed satisfactorily if the first ten terms of infinite series to equation (32) are retained. The behavior of the maximum stress stays steady with the increasing number in terms in (32). Although we can determine the entire the stress field from the coefficients a_n, it is important in fracture mechanics to determine the stress τ_{yz} in the vicinity of the crack tips. τ_{yz} along the crack line can be expressed respectively as

$$\tau_{yz} = -\frac{\mu}{\pi} \sum_{n=1}^{\infty} a_n G_n \int_0^{\infty} [\exp(-sh) + 1] \exp(-\frac{a^2 s^2}{4\beta^2}) J_{2n-1}(sl) \cos(sx) ds \tag{37}$$

For a=0 at x=l, we have the classical stress singularity. However, so long as $a \neq 0$, the semi-infinite integration and the series in the equation (37) are convergent for any variable x. The equation (37) gave a finite stress all along $y = 0$, so there is no stress singularity at the crack tips. At $-l < x < l$, τ_{yz} / τ_0 is very close to unity, and for $x > l$, τ_{yz} / τ_0 possesses finite values diminishing from a maximum value at $x = l$ to zero at $x = \infty$. The semi-infinite numerical integrals, which occur, are evaluated easily by Filon and Simpson methods because the rapid diminution of the integrands. The results are plotted in Figs.2 to7. The following observations can be made:

(I): The maximum stress does not occur at the crack tip, but slightly away from it. This phenomenon has been thoroughly substantiated by Eringen[16] The maximum stress is finite. The distance between the crack tip and the maximum stress point is very small. This distance depends on the lattice parameter, the crack length and the distance between cracks. Contrary to the classical elasticity solution, it is found that no stress singularity is present at the crack tip, and also the present results converge to the classical ones for positions when far away from the crack tip as shown in Fig 4 to Fig 7. **(II):** The anti-plane shear stress at the crack tip becomes infinite as the atomic distance $a \to 0$. This is the classical continuum limit of square root singularity. **(III):** For the $a/\beta =$ constant, viz., the atomic distance does not change, the value of the stress concentrations (at the crack tip) increase higher with the increase of the crack length. Noting this fact, experiments indicate that materials with smaller cracks are more resistant to fracture than those with larger cracks. **(IV):** The stress at the crack tip increases when the distance between cracks increases as shown in Fig 2. This phenomenon is called crack shielding effect. However, the stress at the crack tip increases when the length of the crack increase. **(V):** The stress at the crack tip increases when the lattice parameter decreases

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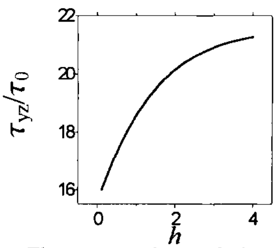

Fig 2. The stress at the crack tip versus h for l=1.0, a/2β=0.0005

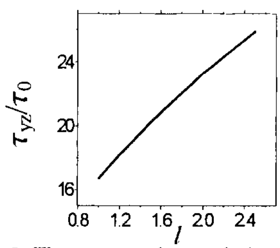

Fig 3. The stress at the crack tip versus l for h=0.3, a/2β=0.0005

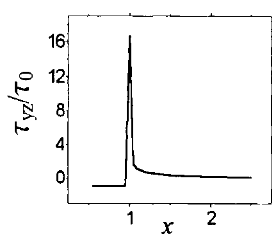

Fig 4. The stress along the crack line versus x for h=0.3, l=1.0, a/2β=0.0005

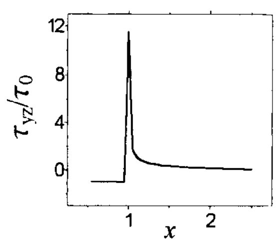

Fig 5. The stress along the crack line versus x for h=0.3, l=1.0, a/2β=0.001

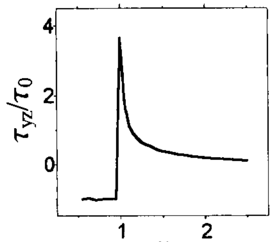

Fig 6. The stress along the crack line versus x for h=0.3, l=1.0, a/2β=0.008

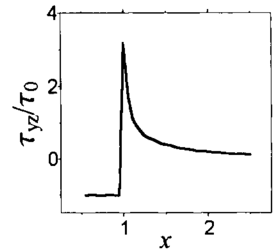

Fig 7. The stress along the crack line versus x for h=0.3, l=1.0, a/2β=0.01

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